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A Remark on Sobolev Spaces. The Case $0 < p < 1$

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1. INTRODUCTION

As is well known, Sobolev spaces are spaces of functions of n variables with suitably defined generalized partial derivatives belonging to L_p, $1 \leqslant p \leqslant \infty$. In this note we shall deal with the extension to the case $0 < p < 1$. The impetus for such an extension originally (cf. [6]) came from the problem of determining the class of functions having a given degree of approximation in certain nonlinear situations; as an example we may mention approximation by rational functions, a question that in our opinion does not yet seem to have obtained a completely adequate treatment (cf. e.g., Lorentz [5, chapter 6] and the references given there). Since we shall mainly be dealing with pathology, it will be sufficient to take $n = 1$. There are several possible definitions of a Sobolev space, which are all equivalent if $1 \leqslant p \leqslant \infty$. Here we shall mainly be dealing with the following one.

DEFINITION 1.1. The Sobolev space W_p^m, $0 < p < 1$, m an integer >0, over the interval $I = [a, b]$ is the (abstract) completion of C^∞, the space of infinitely differentiable functions over I, in the quasinorm:

$$\|f\|_{W_p^m} = (\|f\|_{L_p}^p + \|f'\|_{L_p}^p + \cdots + \|f^{(m)}\|_{L_p}^p)^{1/p}. \tag{1.1}$$

In general $\|g\|_{L_p} = (\int_a^b |g(x)|^p\,dx)^{1/p}$, and $f' = df/dx, \ldots, f^{(m)} = d^m f/dx^m$. It is easy to see that there is a natural mapping $\alpha\colon W_p^m \to L_p$ which is linear and continuous. As was shown years ago by Douady [3], contrary to the case $1 \leqslant p \leqslant \infty$, α fails to be a monomorphism. Here we shall show that actually α has a "retraction," i.e., there exists a continuous linear mapping $\beta\colon L_p \to W_p^m$ such that $\alpha \circ \beta = id =$ the identity mapping, i.e., entirely contradictory to common sense, L_p can be embedded in $W_p{}^m$. In addition, we have $\delta \circ \beta = 0$ where δ: $W_p{}^m \rightarrow W_p^{m-1}$ denotes "derivation." It follows easily that $W_p{}^m$ is isomorphic (as a topological vector space), indeed even isometric, to L_p. Thus by a classical result of Day [2], the dual of $W_p{}^m$ is 0. It follows that $W_p{}^m$ can in no reasonable way be realized even as a space of generalized functions, say, distributions, such spaces being locally convex.

The organization of the note is as follows. Section 2 contains a technical lemma. In Section 3 we prove the above mentioned properties of $W_p{}^m$ in the special case $m = 1$ (including the original counter-example by Douady). In Section 4 we then outline the extension to the case $m > 1$. In Section 5 we consider the dual space. Rather than using the previous results directly we prefer to base our proof here on a general lemma concerning the dual of certain quasi-Banach spaces of functions. Finally, in Section 6 we briefly mention an (unequivalent) less pathological definition of Sobolev space, and in Section 7 we correlate our findings with the point of view of vector-valued functions.

2. A TECHNICAL LEMMA

Let C^m be the space of m times continuously differentiable functions over I. Let $C_+{}^m$ be the space of m times "piecewise continuously differentiable" functions, i.e. $f \in C_+{}^m \Leftrightarrow f \in C^{m-1}$ and $f^{(m-1)}$ has at every point left and right derivatives which have only a finite number of points of discontinuity in I. Thus every $f \in C_+{}^m$ can be represented in the form

$$f = f_0 + \sum_{\nu=1}^{n} c_\nu (x - x_\nu)_+^m, \qquad f_0 \in C^m, \quad c_\nu \text{ real}, \quad x_\nu \in I \quad (\nu = 1,..., n) \quad (2.1)$$

where

$$x_+{}^m = \begin{cases} x^m & \text{if } \; x > 0, \\ 0 & \text{if } \; x \leqslant 0. \end{cases}$$

LEMMA 2.1. *The spaces C^∞ and $C_+{}^m$ have the same completion $W_p{}^m$ in the seminorm* (1.1).

Proof. It suffices to show that C^∞ is dense in $C_+{}^m$. Since it obvious that C^∞ is dense in C^m, it thus also suffices to show that C^m is dense in $C_+{}^m$. In view of the representation (2.1) again it is sufficient to approximate the function $f(x) = x_+{}^m$, $x \in I$, by functions in C^m. Let φ be any function in C^m such that

$$\varphi(x) = \begin{cases} x^m & \text{if } \; x \geqslant 1, \\ 0 & \text{if } \; x \leqslant 0, \end{cases}$$

Set $f_\nu(x) = \nu^{-m}\varphi(\nu x)$, $x \in I$. From $f_\nu^{(j)}(x) = \nu^{j-m}\varphi^{(j)}(\nu x)$ it is easy to see that $f_\nu^{(j)}(x)$ tends to $f^{(j)}(x)$ as ν tends to ∞, uniformly if $0 \leqslant j < m$ and boundedly if $j = m$. In any case we thus have $\|f_\nu^{(j)} - f^{(j)}\|_{L_p} \to 0$ so the desired conclusion $\|f_\nu - f\|_{W_p^m} \to 0$ is obtained.

3. The Case $m = 1$

By Definition 1.1 (with $m = 1$) the elements of W_p^1 are equivalence classes of "fundamental sequences" $\{f_\nu\}_{\nu=1}^\infty$ in C^∞ such that $\|f_\nu - f_\mu\|_{W_p^1} \to 0$ as $\min(\mu, \nu) \to \infty$. (Two fundamental sequences $\{f_\nu\}$ and $\{g_\nu\}$ are in the same class iff $\|f_\nu - g_\nu\|_{W_p^1} \to 0$.) In view of Lemma 2.1 (with $m = 1$) we may as well substitute the class C_+^1 for C^∞. It is plain that if $\{f_\nu\}$ is a fundamental sequence, for the W_p^1-quasinorm, corresponding to $f \in W_p^1$, then $\{f_\nu\}$ and $\{f_\nu'\}$ are fundamental sequences too, for the L_p-quasinorm. Since L_p is complete, it follows that they are convergent in L_p. Let αf and δf be the limits. It is plain that α: $W_p^1 \to L_p$ and δ: $W_p^1 \to L_p$ are linear continuous mappings. Indeed since clearly by (1.1)

$$\|f\|_{W_p^1} = (\|\alpha f\|_{L_p}^p + \|\delta f\|_{L_p}^p)^{1/p}, \tag{3.1}$$

we have

$$\|\alpha f\|_{L_p} \leqslant \|f\|_{W_p^1}, \qquad \|\delta f\|_{L_p} \leqslant \|f\|_{W_p^1},$$

which implies the continuity.

We begin by reproducing the counter-example of Douady (previously also stated in [6]).

Proposition 3.1 (Douady [3]). *The mapping α is* not *a monomorphism.*

Proof. We have to produce a sequence $\{f_\nu\}$ in C_+^1 such that

$$\|f_\nu - f_\mu\|_{W_p^1} \to 0, \qquad \|f_\nu\|_{L_p} \to 0, \qquad \|f_\nu\|_{W_p^1} \not\to 0. \tag{3.2}$$

Then $\{f_\nu\}$ represents an element $0 \neq f \in W_p^1$ with $\alpha f = 0$. Let $\{\eta_\nu\}$ be a sequence of real numbers such that $\nu\eta_\nu$ is decreasing and $\to 0$. We define

$$f(x) = \begin{cases} x, & \text{if } 0 \leqslant x \leqslant 1/\nu, \\ (1/\nu\eta_\nu(1/\nu + \eta_\nu - x), & \text{if } 1/\nu \leqslant x \leqslant 1/\nu + \eta_\nu \end{cases}$$

and extend it by periodicity to the entire real axis and take finally the restriction to I.

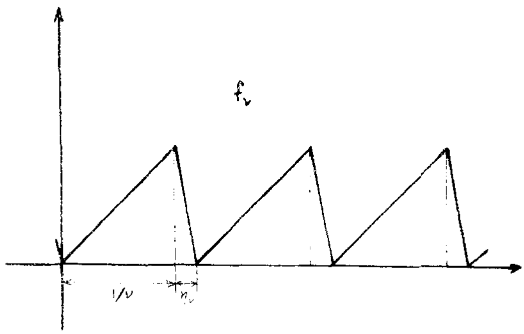

FIGURE 1.

We have

$$f'(x) = \begin{cases} 1 & \text{if } 0 \leqslant x \leqslant 1/\nu, \\ -1/\nu\eta_\nu & \text{if } 1/\nu \leqslant x \leqslant 1/\nu + \eta_\nu . \end{cases}$$

It is now easy to see that

$$\| f_\nu - f_\mu \|_{W_p^1} \leqslant C(1/\nu + 1/\nu\eta_\nu(\nu\eta_\nu)^{1/p}) \qquad \text{if} \quad \mu \geqslant \nu,$$

$$\| f_\nu \|_{L_p} \leqslant C 1/\nu,$$

$$\| f_\nu' \|_{L_p} \geqslant C,$$

where C is a positive constant. Utilizing effectively $0 < p < 1$ we thus see that the requirements (3.2) are met. Thus we get an $f \neq 0$ with $\alpha f = 0$. Indeed we have $\delta f = 1$.

Remark 2.1. Nothing essential happens if we substitute L_p for the Lorentz spaces L_{pq}, $0 < p < 1$, $0 < q \leqslant \infty$ (leading to Lorentz–Sobolev spaces W_{pq}^m rather than Sobolev spaces W_p^m). However if we take the space L_{1q} the construction of Proposition 3.1 breaks down even if $q > 1$. The case $q = \infty$ might be particularly interesting. It is known that $L_{1\infty}$ is to some extent less pathological than L_p, $0 < p < 1$, so maybe this will be the case too for W_1^m. E.g., it is known (Haaker [4], Cwikel and Sagher [1]) that the dual of $L_{1\infty}$ is not 0. The space $L_{1\infty}$ (sometimes known as the Marcinkiewicz space) appears also in classical analysis: It is known that the L_1 image of many classical operators (Hardy–Littlewood maximal operator, Hilbert transform etc.) is contained in $L_{1\infty}$.

Now we proceed to the retraction β of α mentioned in Section 1.

PROPOSITION 3.2. *There exists a continuous linear mapping* $\beta: L_p \to W_p^1$ *such that* $\alpha \circ \beta = id$, $\delta \circ \beta = 0$.

Proof. Let S be the space of "simple" functions (i.e., $g \in S \Leftrightarrow g$ has at most a finite number of would-be discontinuities ξ_i $(i = 0,..., N)$, $a = \xi_0 < \xi_1 < \cdots < \xi_N = b$, and is constant in the intervals between these (ξ_{i-1}, ξ_i) $(i = 1,..., N)$). Since S is dense in L_p it suffices to construct βg for $g \in S$. A sequence $\{f_\nu\}$ is called a *regular* fundamental sequence for $g \in S$ if $f_\nu \in C_+^1$ is of the following type: f is constant except in "small" intervals containing the would-be discontinuities. If $\xi = \xi_i \in I$ is a would-be discontinuity of g, f_ν is taken to be linear on some small interval $J = J_i$ with center ξ_i, say, of length $a = a_{i\nu}$. It is required of course that $a_{i\nu} \to 0$ as $\nu \to \infty$ (which is the precise qualification of "small")

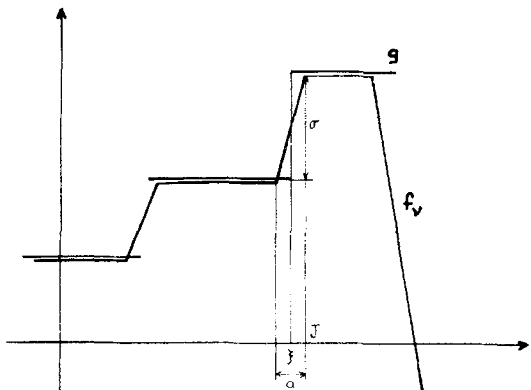

FIGURE 2.

We have $f_\nu' = \sigma/a$ throughout J, if σ denotes the size of the jump of g at ξ. The contribution to $\| f_\nu' \|_{L_p}$ of the point ξ is thus of the magnitude $1/a \cdot a^{1/p}$. Utilizing again $0 < p < 1$ we see that $\| f_\nu - f_\mu \|_{W_p^1} \to 0$ so that $\{f_\nu\}$ is effectively a fundamental sequence. It is also plain that two regular fundamental sequences are equivalent. Thus to each $g \in S$ we have associated a unique element $f = \beta g \in W_p^1$. It is obvious that $\| \beta g \|_{W_p^1} = \| g \|_{L_p}$. It is likewise perfectly obvious that $\alpha(\beta g) = g$ and $\delta(\beta g) = 0$. The only thing that remains is to check that β is linear. To this end let $g = g^1 + g^2$ and pick up two regular fundamental sequences $\{f_\nu^1\}$ and $\{f_\nu^2\}$ for g^1 and g^2, respectively, choosen in such a way that the would-be discontinuities ξ_i are the same and also the small intervals $J_{\nu i}$ around ξ_i. Then it is plain that $\{f_\nu^1 + f_\nu^2\}$ is a regular fundamental sequence for g. This proves additivity $\beta(g^1 + g^2) = \beta g^1 + \beta g^2$. Homogeneity $\beta(\lambda g) = \lambda \beta g$, on the other hand, is obvious. The proof is complete.

Next we state the consequences of Proposition 3.2.

COROLLARY 3.1. *We have $W_p^1 \approx L_p \oplus L_p$. An isomorphism is provided by $f \mapsto (\beta\alpha f, \delta f)$. We have $\delta f = 0$ iff f is of the form $f = \beta g$.*

Proof. This follows at once if we invoke the following general algebraic lemma. (We note that (3) is fulfilled in view of (3.1).)

LEMMA 3.1. *Let X, Y, Z be vector spaces and let $\alpha\colon X \to Y$, $\beta\colon Y \to Z$, $\delta\colon X \to Z$ be linear mappings such that:*

(1) $\alpha \circ \beta = id$,

(2) $\delta \circ \beta = 0$,

(3) $\alpha \oplus \delta$ *is a monomorphism,*

(4) δ *is an epimorphism.*

Then $X = \operatorname{Im} \beta \dotplus \operatorname{Ker} \alpha$. *In addition* $\operatorname{Im} \beta \approx Y$, $\operatorname{Ker} \alpha \approx Z$ *so that* $X \approx Y \oplus Z$. *Also* $\operatorname{Im} \beta = \operatorname{Ker} \delta$. *If X, Y, Z are topological vector spaces and α, β, δ continuous then the isomorphisms and the direct sums are topological.*

(Here $\dotplus$ stands for the direct sum $=$ linear hull, and $\oplus$ for the abstract direct sum.)

Proof. Let $f \in X$. Then $f = f_1 + f_2$ with $f_1 = \beta\alpha f$, $f_2 = f - \beta\alpha f$. It is clear that $f_1 \in \operatorname{Im} \beta$. On the other hand since $\alpha f_2 = \alpha f - \alpha\beta\alpha f = \alpha f - \alpha f = 0 - 0 = 0$ we have also $f_2 \in \operatorname{Ker} \alpha$. Conversely if $f = f_1 + f_2$ with $f_1 \in \operatorname{Im} \beta$, $f_2 \in \operatorname{Ker} \alpha$ we find $\alpha f = \alpha f_1$, $\beta\alpha f = \beta\alpha f_1 = f_1$ and $f_2 = f - f_1 = f - \beta\alpha f$. Thus we have proven $X = \operatorname{Im} \beta \dotplus \operatorname{Ker} \alpha$. Since β evidently is a monomorphism we have $\operatorname{Im} \beta \approx Y$. Uptil now we only used (1). Let now $f \in \operatorname{Ker} \alpha$. Then by (3) we cannot have $\delta f = 0$ unless $f = 0$. I.e., $\delta \mid \operatorname{Ker} \alpha$ is a monomorphism. On the other hand, since in view of (2) $\delta \operatorname{Im} \beta = 0$, it is by (4) an epimorphism. Thus $\operatorname{Ker} \alpha \approx Z$. There remains the relation $\operatorname{Im} \beta = \operatorname{Ker} \delta$. By (2) $\operatorname{Im} \beta \subset \operatorname{Ker} \delta$ is obvious. To prove $\supset$ let thus $\delta f = 0$. Write again $f = f_1 + f_2$ with $f_1 = \beta\alpha f, f_2 = f - \beta\alpha f$. Then $\alpha f_2 = \alpha f - \alpha\beta\alpha f = \alpha f - \alpha f = 0$, $\delta f_2 = \delta f - \delta\beta\alpha f = 0$. Therefore, $f_2 = 0$ and $f = f_1 = \beta\alpha f \subset \operatorname{Im} \beta$.

COROLLARY 3.2. *We have $W_p^1 \approx L_p$.*

Proof. This follows at once from Corollary 3.1 using the well-known fact that $L_p \oplus L_p \approx L_p$.

4. EXTENSION TO THE CASE $m > 1$

We now very quickly sketch an extension of the results of Section 3 to the case $m > 1$.

We consider the space W_p^m. By Definition 1.1 (now with $m > 1$) the elements of W_p^m are again equivalence classes of fundamental sequences $\{f_\nu\}_{\nu=1}^\infty$ in C^∞ such that $\|f_\nu - f_\mu\|_{W_p^m} \to 0$. By Lemma 2.1 we may replace C^∞ by C_+^m. The mappings α: $W_p^m \to L_p^p$ and δ: $W_p^m \to W_p^{m-1}$ are defined in an analogous fashion as in the case $m = 1$. In particular we have corresponding to (3.1)

$$\|f\|_{W_p^m} = (\|\alpha f\|_{L_p}^p + \|\delta f\|_{W_p^{m-1}}^p)^{1/p}. \tag{4.1}$$

We have not attempted to generalize directly the Douady counter-example (Proposition 3.1). Instead we proceed at once to the generalization of Proposition 3.2.

PROPOSITION 4.1. *There exists a continuous linear mapping* β: $L_p \to W_p^m$ *such that* $\alpha \circ \beta = id$, $\delta \circ \beta = 0$.

Proof (*Outline*). The only difficulty, compared with the case $m = 1$, is to find an adequate notion of regular fundamental sequence for $g \in S$. For simplicity we take $g = 1_+$ (i.e., x_+^m with $m = 0$!) and in order not to complicate the notation $m = 3$. We choose the "small" interval J with center $\xi = 0$ of length $a + 2b + 4c$ for some $a = a_\nu$, $b = b_\nu$, $c = c_\nu$. We also assume $a \gg b \gg c$. We choose $f = f_\nu$ in such a way that f''' is, with some $A = A_\nu$, successively counted from left to right A, 0, $-A$, 0, $-A$, 0, A on intervals of length c, b, c, a, c, b, c, respectively.

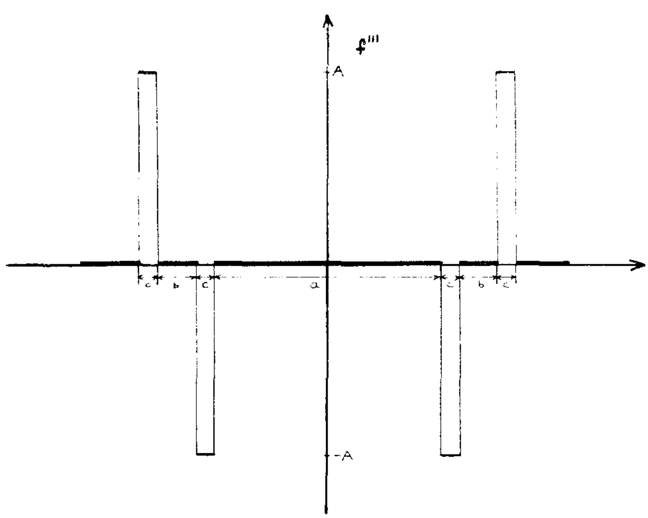

FIGURE 3.

In order to make f match with 1_+ we have to require that $Aabc \doteqdot 1$. On the other hand the contribution of f''', f'', f' to $\|f\|_{W_p^3}$ is of the orders $Ac^{1/p}$, $Acb^{1/p}$, $Abca^{1/p}$, respectively. Thus we have to require that $c^{1/p-1}a^{-1}b^{-1} \to 0$, $b^{1/p-1}a^{-1} \to 0$, $a^{1/p-1} \to 0$, a requirement which obviously can be met, since $0 < p < 1$. It is now obvious that the whole of the argument of Proposition 2.2 can be carried over.

COROLLARY 4.1. *We have* $W_p^m \approx L_p \oplus W_p^{m-1}$.

Proof. Lemma 3.1 is still applicable.

COROLLARY 4.2. *We have* $W_p^m \approx L_p$.

Proof. Induction over m together with $L_p \approx L_p \oplus L_p$.

5. THE DUAL SPACE

PROPOSITION 5.1. *The dual of* W_p^m *is* 0.

Proof. This follows immediately from Corollary 4.2 since by the well-known result of Day [2] the dual of L_p is 0 (if $0 < p < 1$). We prefer however to give a direct proof which does not use neither Day's theorem nor Corollary 4.2 (in full). We first remark that it clearly suffices to prove the same result for $\mathring{W}_p^m$, the Sobolev space of periodic, say, 2π-periodic functions, which is defined in an analogous way as in Definition 1.1. This can be seen as follows. We may clearly assume that $b - a < 2\pi$. Let ρ: $\mathring{W}_p^m \to W_p^m$ be the restriction map. Clearly ρ is an epimorphism. If λ is a continuous linear functional on W_p^m then $\mathring{\lambda} = \lambda \circ \rho$ is a continuous linear functional on $\mathring{W}_p^m$. So if we know that $\mathring{\lambda} = 0$ by necessity, it follows that $\lambda = 0$. Next we invoke the following general lemma, which might have some interest of its own.

LEMMA 5.1. *Let E be a quasinormed space of periodic functions. Assume that* (1) $\mathring{C}^\infty$ *is a dense subset of E, the injection of* $\mathring{C}^\infty$ *into E being continuous,* (2) *E is translation invariant, the quasinorm being translation invariant too,* (3) *E is invariant for multiplication with* $\mathring{C}^\infty$ *functions. If there exists at least one continuous linear functional* $\lambda \neq 0$ *in E then* $\lambda(f) = \int_0^{2\pi} f(x)\, dx$ *must be a continuous linear functional too. In other words, there holds the inequality*:

$$\left| \int_0^{2\pi} f(x)\, dx \right| \leqslant C \|f\|_E \qquad (f \in \mathring{C}^\infty). \tag{5.1}$$

In our case (5.1) *gives*

$$\left| \int_0^{2\pi} f(x)\, dx \right| \leqslant C \| f \|_{W_p^m} \qquad (f \in \mathring{C}^\infty).$$

Using now the approximation device of the proof of Proposition 4.1 (indicated for $m = 3$) we see that

$$\left| \int_0^{2\pi} f(x)\, dx \right| \leqslant C \| f \|_{L_p} \qquad (f \in \mathring{S}),$$

which obviously cannot be true if $0 < p < 1$. (Take f to be a characteristic function of an interval of length l and let l tend to 0.) The (alternative) proof of Proposition 5.1 is complete.

There remains the proof.

Proof of Lemma 5.1. Denoting the dual space by E', if $0 \neq \lambda \in E'$ then we must have $\lambda(e^{inx}) \neq 0$ for some integer n. Thus upon replacing λ by $\tilde{\lambda}$ defined by $\tilde{\lambda}(f) = \lambda(e^{inx} f)$ we may assume that $\lambda(1) \neq 0$, say, $\lambda(1) = c$. (Here we have invoked condition (3).) We may also assume that $\| \lambda \| = 1$. Next we replace λ by $\bar{\lambda}$ defined by $\bar{\lambda}(f) = 1/2\pi \int_0^{2\pi} \lambda(f_h)\, dh$ (where $f_h(x) = f(x + h)$ denotes the translation of $f(x)$). We see that we may assume that λ is translation invariant too (now condition (2) has been used!). But the restriction of λ to $\mathring{C}^\infty$ is a (periodic) distribution and the only translation invariant distributions are the constant functions. Hence we get

$$\lambda(f) = c \int_0^{2\pi} f(x)\, dx \qquad (f \in \mathring{C}^\infty)$$

and (5.1) follows with $C = c^{-1}$.

Remark 5.1. Note that Lemma 5.1 applied to $E = L_p$ incidentally gives a new proof of Day's theorem. (This is of course nothing but the proof of Proposition 5.1 in the limiting case $m = 0$.) The same argument applies to L_{pq} if $0 < p < 1$, $0 < q < \infty$ (cf. Haaker [4]).

6. A Less Pathological Definition

If $1 \leqslant p \leqslant \infty$ there are several alternative equivalent definitions of Sobolev space. If $0 < p < 1$ (if they generalize) they need not be equivalent anymore. In the preceeding Sections we have used one such definition (Definition 1.1). This has lead us to pathology only. Therefore we now suggest still another one. For simplicity we state it for the case $m = 1$ only.

DEFINITION 6.1. $\mathscr{W}_p^1$ is the space of functions $f \in L_p$ which are "differentiable in quasinorm," i.e., there exists a function $f' \in L_p$ such that

$$\int_{I \cap I_{-h}} \left| \frac{f(x+h) - f(x)}{h} - f'(x) \right|^p dx \to 0,$$

where $I_h = [a+h, b+h]$ is the translation of I by h. We equip $\mathscr{W}_p^1$ with the following quasinorm:

$$\| f \|_{\mathscr{W}_p^1} = \| f \|_{W_p^1} + \sup_h \left(\int_{I \cap I_{-h}} | f(x+h) - f(x) |^p \, dx \right)^{1/p} \Big/ | h |.$$

It is now a simple excercise of functional analysis to show that $\mathscr{W}_p^1$ indeed is complete. Moreover $\mathscr{W}_p^1$ is continuously embedded in L_p. In [6] some (mostly fragmentary, though) results for related space Lip_p^θ of Lipschitz continuous in the L_p quasinorm functions were obtained. In particular it was shown that Lip_p^θ is embedded in L_1, if $\theta + 1 > 1/p$ and $0 < \theta \leqslant 1$ (the latter is probably a merely technical restriction). The proof adapted to the present case shows that $\mathscr{W}_p^1$ can be imbedded in L_1, if $p > \frac{1}{2}$. It is conceivable that $p > \frac{1}{2}$ is the right bound. (Indeed we have reasons to suspect that the dual of $\mathscr{W}_p^1$ is 0 if $0 < p \leqslant \frac{1}{2}$.)

7. THE POINT OF VIEW OF DIFFERENTIABLE FUNCTIONS

Let A be any quasi-Banach space (i.e., a complete topological vector space the topology of which comes from a quasinorm). We consider periodic functions φ with values in A. (For simplicity we consider the periodic case only.) The space of all continuous periodic functions φ is denoted by $\mathring{C}(A)$. It is a quasi-Banach space for the quasinorm,

$$\| \varphi \|_{\mathring{C}(A)} = \sup_h \| \varphi(h) \|_A .$$

We next introduce the space $\mathring{C}^1(A)$ as the completion of $\mathring{C}^\infty(A)$ in the quasinorm

$$\| \varphi \|_{\mathring{C}^1(A)} = \| \varphi \|_{\mathring{C}(A)} + \| \varphi' \|_{\mathring{C}(A)} .$$

If $A = \mathring{L}_p$ the assignment $f \mapsto \varphi_f(h) = f_h$ (translation) defines an embedding of $\mathring{W}_p^1$ into $\mathring{C}^1(\mathring{L}_p)$. The Douady counter-example (Proposition 3.1) can now be rephrased as follows: *The space $\mathring{C}^1(A)$ cannot in general be embedded in $C^0(A)$.* (There prompts now the question if this is the case for other spaces than $A = \mathring{L}_p$, cf. Remark 4.1.) On the other hand (cf. Definition 5.1) if we introduce $\mathring{\mathscr{C}}^1(A)$ as the space of continuously differentiable functions, equiped with the quasinorm

$$\| \varphi \|_{\mathring{\mathscr{C}}^1(A)} = \| \varphi \|_{C^1(A)} + \sup_{h,k} \| \varphi(h + k) - \varphi(h) \|_A / | k |,$$

we obtain a quasi-Banach space which is continuously embedded in $\mathring{C}(A)$.

Summary

It is shown that the (conveniently defined) Sobolev space W_p^m, m integer >0, $0 < p < 1$, is isomorphic to L_p. Consequently its dual is 0.

Note added in Proof. The best generalization of Sobolev spaces to $0 < p < 1$ is however obtained by substituting for L_p the Hardy class H_p. Then practically all the usual properties of Sobolev (and Besov) spaces for $1 < p < \infty$ carry over to the full range $0 < p < \infty$. This is possible by virtue of the real variable characterization of H_p due to Fefferman-Stein (Acta Math. **129** (1972), 137–193). See e.g. my lecture notes "New thoughts on Besov spaces" (hopefully to be published at the Duke University Press).

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